ELSEVIER Journal of Geometry and Physics 26 (1998) 51–78

JOURNAL OF GEOMETRY AND PHYSICS

Conformal invariant functionals of immersions of tori into $\mathbb{R}^3$

P.G. Grinevich [a,*], M.U. Schmidt [b,1]

[a] L.D. Landau Institute for Theoretical Physics, Kosygina 2, Moscow 177940, Russia
[b] Institut fur Theoretische Physik, Freie Universität, Berlin, Arnimallee 14, D-14195 Berlin, Germany

Received 6 May 1997
Communicated by M. Modugno

Abstract

We show, that higher analogs of the Willmore functional, defined on the space of immersions $M^2 \to \mathbb{R}^3$, where M^2 is a two-dimensional torus, $\mathbb{R}^3$ is the three-dimensional Euclidean space are invariant under conformal transformations of $\mathbb{R}^3$. This hypothesis was formulated recently by I.A. Taimanov.

Higher analogs of the Willmore functional are defined in terms of the Modified Novikov–Veselov hierarchy. This soliton hierarchy is associated with the zero-energy scattering problem for the two-dimensional Dirac operator.

Subj. Class.: Differential geometry; Strings
1991 MSC: 53C42, 53A10, 35P05, 35Q58
Keywords: Generalized Willmore functionals; Dirac operator; Conformal invariants; Bloch variety

1. Introduction

To start with, we would like to recall the following interesting fact from the theory of two-dimensional surfaces in $\mathbb{R}^3$ (see [20, p. 110] and references therein). Let $X : M^2 \to \mathbb{R}^3$ be a smooth immersion of a compact orientable surface M^2 into the Euclidean space $\mathbb{R}^3$

* Corresponding author. E-mail: grinev@physik.fu-berlin.de. This work was fulfilled during the author's visit to the Freie Universität, Berlin, Germany, which was supported by the Humboldt-Foundation. This work was also partially supported by Russian Foundation for Basic Research, Grant No. 95-01-755.

[1] E-mail: mschmidt@omega.physik.fu-berlin.de. Supported by the DFG, SFB 288 "Differentialgeometrie und Quantenphysik".

PII S0393-0440(97)00038-7

(i.e., a smooth map from M^2 to $\mathbb{R}^3$ such that its Jacobi matrix is non-degenerate everywhere on M^2, but the image is allowed to self-intersect). Let T be the Willmore functional

$$T = \int\limits_{M^2} H^2 \,\mathrm{d}S, \tag{1}$$

where H denotes the mean curvature, $\mathrm{d}S$ is the volume element on M^2 generated by the immersion. Then T is invariant under conformal transformations of $\mathbb{R}^3$.

It is natural to pose the problem of constructing other conformal invariant functional of immersions.

It is well-known, that many constructions from the soliton theory have natural analogs in geometry and vice versa. In particular, important information on how to study immersions of two-dimensional surfaces into $\mathbb{R}^3$ using soliton methods has been provided [1] by A.I. Bobenko. Any immersed surface possesses (at least locally) a conformal coordinate system (see for example [3, p. 110]), i.e. a coordinate system such that $\mathrm{d}s^2 = f(z,\bar{z})\,\mathrm{d}z\,\mathrm{d}\bar{z}$. In conformal coordinates this immersion can be locally represented by the Generalized Weierstrass Formulas (see Section 2) and the potential $U(z,\bar{z})$ is uniquely defined. In [17] it was shown, that any analytic immersion of a compact orientable two-dimensional manifold into $\mathbb{R}^3$ can be globally represented by these formulas.

The generalized Weierstrass formulas are based on the zero-energy eigenfunctions of the two-dimensional Dirac operator with a real potential $U(z,\bar{z})$. The zero-energy spectral problem for this operator arose in the soliton theory as an auxiliary linear problem for the hierarchy of the Modified Novikov–Veselov equations (MNV) (see [2]). These nonlinear integrable equations with two spatial variables, introduced by Bogdanov, have infinitely many conservation laws.

Konopelchenko and Taimanov [9] showed that the quadratic MNV conservation law

$$H_1 = 4\int\int U^2(z,\bar{z})\,\mathrm{d}x \wedge \mathrm{d}y,\ z = x + \mathrm{i}y \tag{2}$$

coincides with the Willmore functional T. Taimanov [18] formulated a hypothesis that all higher MNV conservation laws also generate functionals on immersions of closed orientable two-dimensional surfaces into $\mathbb{R}^3$, invariant under conformal transformations of $\mathbb{R}^3$. He also did some numerical experiments with surfaces of revolution, confirming this assumption.

During his visit to the Freie Universität, Berlin, in September–October 1996, Taimanov attracted the authors attention to this problem. In the present text we prove this hypothesis for immersions of tori into $\mathbb{R}^3$.

Of course, it would be natural to extend this result to immersions of higher genus surfaces into $\mathbb{R}^3$. But here we meet the following problem. In contrast with H_1 the higher conservation laws are non-local in terms of the potential. For double-periodic potentials they are defined in terms of the zero-energy dispersion curve (the Riemann surface of the zero-energy Bloch function). If the genus is greater than one, we have to study modular invariant Dirac operators in the Lobachevskian plane with a non-Abelian group of translations. The corresponding Bloch theory has not been constructed until now, thus we could not define the higher conservation laws. It would be interesting to develop the corresponding Bloch theory, but this problem looks rather non-trivial. Thus, we restrict ourself to the genus one case only, where all integrals of motion are well-defined.

The idea of our proof is the following. We show that infinitesimal conformal transformations of $\mathbb{R}^3$ correspond to infinitesimal Darboux transformations of the Dirac operator (infinitesimal dressings with degenerate kernels). It is convenient to express these deformations in terms of Cauchy–Baker–Akhiezer kernels, introduced by Orlov and Grinevich in [6] (see also the review [7]). From the explicit formulas for such deformations it follows, that the deformed zero-energy Bloch function is meromorphic on the same Riemann surface as the original one. Recalling, that this Riemann surface completely determines all conservation laws, we complete the proof.

Remark 1. An alternative proof of the theorem, namely that conformal transformations do not change the zero-energy Bloch variety, was obtained by Pinkall (private communication). He calculated the action of finite conformal transformations on the Bloch function using a quaternionic representation of the generalized Weierstrass formulas as suggested by Kamberov et al. [8].

Remark 2. Finite Darboux transformations for the Dirac operator are discussed in the book by Matveev and Salle [13], Laplace transformations for the Dirac operator are discussed by Ferapontov [5].

2. Generalized Weierstrass construction

Let L be the two-dimensional Dirac operator

$$L = \begin{bmatrix} \partial_z & -U(z,\bar{z}) \\ U(z,\bar{z}) & \partial_{\bar{z}} \end{bmatrix} \tag{3}$$

with a real potential $U(z,\bar{z})$. Let $\Psi(z,\bar{z})$ be a zero-energy solution of the Dirac equation

$$L\Psi(z,\bar{z}) = 0,\ \Psi(z,\bar{z}) = \begin{pmatrix} \Psi_1(z,\bar{z}) \\ \Psi_2(z,\bar{z}) \end{pmatrix}. \tag{4}$$

Then the *generalized Weierstrass formulas* (see [17] and refereces therein)

$$\begin{aligned} X_1(z,\bar{z}) + \mathrm{i}X_2(z,\bar{z}) &= C_1 + \mathrm{i}C_2 + \mathrm{i}\int_{z_0}^{z} (\bar{\Psi}_1^2(z',\bar{z}')\,\mathrm{d}z' - \bar{\Psi}_2^2(z',\bar{z}')\,\mathrm{d}\bar{z}'), \\ X_1(z,\bar{z}) - \mathrm{i}X_2(z,\bar{z}) &= C_1 - \mathrm{i}C_2 + \mathrm{i}\int_{z_0}^{z} (\Psi_2^2(z',\bar{z}')\,\mathrm{d}z' - \Psi_1^2(z',\bar{z}')\,\mathrm{d}\bar{z}'), \\ X_3(z,\bar{z}) &= C_3 - \int_{z_0}^{z} (\Psi_2(z',\bar{z}')\bar{\Psi}_1(z',\bar{z}')\,\mathrm{d}z' + \Psi_1(z',\bar{z}')\bar{\Psi}_2(z',\bar{z}')\,\mathrm{d}\bar{z}'), \end{aligned} \tag{5}$$

defines a map of the plane $\mathbb{R}^2$ to the Euclidean space $\mathbb{R}^3$. In (5), z_0 is a fixed point in the z-plane and the integrals are taken over some path connecting the points z_0 and z. From (4) it follows that the integrands in (5) are closed forms, thus the map does not depend on a specific choice of the path. Here C_1, C_2, C_3 are arbitrary real constants.

The formulas (5) are equivalent to:

$$\mathrm{d}[\sigma^2 X_1 + \sigma^1 X_2 - \sigma^3 X_3] = \begin{bmatrix} \bar{\Psi}_1(z,\bar{z}) & \bar{\Psi}_2(z,\bar{z}) \\ -\Psi_2(z,\bar{z}) & \Psi_1(z,\bar{z}) \end{bmatrix} \begin{bmatrix} 0 & \mathrm{d}z \\ \mathrm{d}\bar{z} & 0 \end{bmatrix} \begin{bmatrix} \Psi_1(z,\bar{z}) & -\bar{\Psi}_2(z,\bar{z}) \\ \Psi_2(z,\bar{z}) & \bar{\Psi}_1(z,\bar{z}) \end{bmatrix}, \tag{6}$$

where $\sigma^1, \sigma^2, \sigma^3$ are the standard Dirac matrices

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma^2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \qquad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{7}$$

The generalized Weierstrass map is conformal, i.e. the metric $\mathrm{d}\tilde{s}^2$ on $\mathbb{R}^2$ induced by this map is proportional to the standard one: $\mathrm{d}\tilde{s}^2 = g(z,\bar{z})\,\mathrm{d}z\,\mathrm{d}\bar{z}$.

Assume that we have a map of a two-dimensional torus into $\mathbb{R}^3$. Then the corresponding potentials $U(z,\bar{z})$ is periodic

$$U(z+\bar{T}_1, \bar{z}+T_1) = U(z+T_2, \bar{z}+\bar{T}_2) = U(z,\bar{z}). \tag{8}$$

Also the eigenfunction $\boldsymbol{\Psi}(z,\bar{z})$ is periodic or anti-periodic, i.e.

$$\begin{aligned} &\boldsymbol{\Psi}(z+T_1, \bar{z}+\bar{T}_1) = \mathcal{W}_1 \boldsymbol{\Psi}(z,\bar{z}), \\ &\Psi(z+T_2, \bar{z}+\bar{T}_2) = \mathcal{W}_2 \boldsymbol{\Psi}(z,\bar{z}), \quad \mathcal{W}_1^2 = \mathcal{W}_2^2 = 1. \end{aligned} \tag{9}$$

The coordinate z is defined up to linear transformations $z \to az + b$, a and $b \in \mathbb{C}$, $a \neq 0$. Thus, without loss of generality we may assume

$$T_1 = 1, \qquad T_2 = \tau, \quad \operatorname{Im}\tau > 0. \tag{10}$$

Conditions (8) and (9) are necessary, but, of course, not sufficient for periodicity of the generalized Weierstrass map. Necessary and sufficient conditions for periodicity can be formulated in terms of the Bloch variety. They are obtained in a forthcoming paper by Taimanov and Schmidt . We do not use these conditions in our text, thus we will not discuss them in further details.

3. Bloch function and Bloch variety

In this section we assume that $U(z,\bar{z})$ is real, smooth, and double-periodic (8). With any such potential we associate a one-dimensional subvariety Γ in the two-dimensional complex space $(\mathbb{C}\backslash 0)^2$.

The first object we need is the *Bloch function*. By definition, the Bloch functions $\psi(w_1, w_2, z, \bar{z})$ are quasi-periodic solutions of the Dirac equation (4) with the following periodicity properties:

$$\begin{aligned}\psi(w_1, w_2, z+1, \bar{z}+1) &= w_1\psi(w_1, w_2, z, \bar{z}),\\ \psi(w_1, w_2, z+\tau, \bar{z}+\bar{\tau}) &= w_2\psi(w_1, w_2, z, \bar{z}).\end{aligned} \tag{11}$$

The pairs of multipliers w_1, w_2 possessing at least one non-zero Bloch solution form a complex one-dimensional subvariety $\Gamma \in (\mathbb{C}\backslash 0) \times (\mathbb{C}\backslash 0)$ (see [12]). This variety is called the *Bloch variety* or the *zero-energy dispersion curve*. For a generic potential $U(z, \bar{z})$ the genus of Γ is infinite.

The Bloch functions form a one-dimensional holomorphic bundle over Γ (it is shown below that for generic $\lambda \in \Gamma$, a Bloch solution is unique up to normalization). It is convenient to fix a section of this bundle $\psi(\lambda, z, \bar{z})$, by assuming

$$\psi_1(\lambda, z, \bar{z}) + \psi_2(\lambda, z, \bar{z})|_{z=z_1} = 1, \tag{12}$$

where z_1 is an arbitrary fixed point.

The logarithms of the multipliers $w_1(\lambda)$, $w_2(\lambda)$

$$p_1(\lambda) = \frac{1}{\mathrm{i}} \ln w_1(\lambda), \qquad p_2(\lambda) = \frac{1}{\mathrm{i}|\tau|} \ln w_2(\lambda), \tag{13}$$

are called *quasi-momentum functions*. Of course, they have non-trivial increments while going along cycles in Γ, and they are defined up to adding $2\pi n_1$, $2\pi n_2/|\tau|$, respectively, where n_1 and n_2 are some integers. Thus, the functions Im $p_1(\lambda)$, Im $p_2(\lambda)$ are single-valued in Γ. The *differentials of the quasi-momentum functions*

$$\mathrm{d}p_1(\lambda) = \frac{\partial}{\partial\lambda} p_1(\lambda)\,\mathrm{d}\lambda, \qquad \mathrm{d}p_2(\lambda) = \frac{\partial}{\partial\lambda} p_2(\lambda)\,\mathrm{d}\lambda, \tag{14}$$

are single-valued and holomorphic on the finite part of Γ.

In our text the Dirac operator (3) is symmetric and the potential $U(z, \bar{z})$ is real. Let us show, that the corresponding Bloch variety Γ possesses $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a group of symmetries. An analogous statement for the fixed-energy Bloch variety corresponding to a two-dimensional self-adjoint Schrödinger operator was proved in [10]. The proof from [10] may be applied to (3) after a minimal modification.

The operator (3) with real potential $U(z, \bar{z})$ has the following symmetry. If $\psi(w_1, w_2, z, \bar{z})$ is a Bloch solution of (4), then the function

$$\psi^{\dagger}(w_1, w_2, z, \bar{z}) = \begin{pmatrix} \bar{\psi}_2(w_1, w_2, z, \bar{z}) \\ -\bar{\psi}_1(w_1, w_2, z, \bar{z}) \end{pmatrix} \tag{15}$$

is also a Bloch solution of (4) with multipliers $\bar{w}_1$ and $\bar{w}_2$ respectively. Thus, the surface Γ possesses an antiholomorphic involution (we denote it by $\sigma\theta$ for historical reasons)

$$\sigma\theta : \Gamma \to \Gamma, \qquad \sigma\theta : (w_1, w_2) \to (\bar{w}_1, \bar{w}_2), \tag{16}$$

and

$$\psi^{\dagger}(\lambda, z, \bar{z}) = n^{-1}(\lambda)\psi(\sigma\theta(\lambda), z, \bar{z}), \tag{17}$$

where $n(\lambda)$ is a scalar function, meromorphic in λ and independent on $z, \bar{z}$.

It is less trivial to see that the surface Γ possesses a holomorphic involution:

$$\sigma : \Gamma \to \Gamma, \qquad \sigma : (w_1, w_2) \to (w_1^{-1}, w_2^{-1}). \tag{18}$$

To prove it, let us fix a generic point $\lambda \in \Gamma$. Denote by $\mathcal{L}_{w_1 w_2}$ the Banach space of all locally square-integrable two-component complex-valued vector-functions on $\mathbb{R}^2$ with the periodicity properties (11). The space $\mathcal{L}_{w_1^{-1} w_2^{-1}}$ is naturally dual to $\mathcal{L}_{w_1 w_2}$. Namely, if $\boldsymbol{f}(z, \bar{z}) \in \mathcal{L}_{w_1 w_2}$ and $\boldsymbol{g}(z, \bar{z}) \in \mathcal{L}_{w_1^{-1} w_2^{-1}}$, then we define a scalar product by

$$\langle f, g \rangle = \int_0^1 \int_0^1 \mathrm{d}t_1 \wedge \mathrm{d}t_2 [f_1(t_1 + \tau t_2, t_1 + \bar{\tau} t_2) g_1(t_1 + \tau t_2, t_1 + \bar{\tau} t_2) + f_2(t_1 + \tau t_2, t_1 + \bar{\tau} t_2) g_2(t_1 + \tau t_2, t_1 + \bar{\tau} t_2)]. \tag{19}$$

Let $\boldsymbol{f}^{(0)} = \psi(w_1, w_2, z, \bar{z}), \boldsymbol{f}^{(1)}, \ldots, \boldsymbol{f}^{(n)}, \ldots$ be the Jordan basis for the Dirac operator L in the space $\mathcal{L}_{w_1 w_2}$. Also let $\boldsymbol{g}^{(0)}, \boldsymbol{g}^{(1)}, \ldots, \boldsymbol{g}^{(n)}, \ldots$ be the dual basis in $\mathcal{L}_{w_1^{-1} w_2^{-1}}$. The functions $\boldsymbol{g}^{(n)}$ form a Jordan basis for the transposed operator L^T. But L is symmetric with respect to this scalar product, thus it has a zero eigenfunction in the space $\mathcal{L}_{w_1^{-1} w_2^{-1}}$. Hence if $(w_1, w_2) \in \Gamma$, then $\sigma(w_1, w_2) = (w_1^{-1}, w_2^{-1}) \in \Gamma$.

One of the main properties of the Bloch variety Γ is the following: Γ may be treated as a *complete set of integrals of motion* for the *Modified Novikov–Veselov hierarchy*.

Indeed, consider the space of all real-valued smooth double-periodic functions on $\mathbb{C}^1 = \mathbb{R}^2$ with a fixed pair of periods: 1 and τ. The modified Novikov–Veselov hierarchy (MNV) (see Section 4) defines an infinite collection of flows on this space

$$\frac{\partial U(z, \bar{z}, t_{2n+1})}{\partial t_{2n+1}} = K_{2n+1}[U] + \bar{K}_{2n+1}[U], \tag{20}$$

$$\frac{\partial U(z, \bar{z}, \tilde{t}_{2n+1})}{\partial \tilde{t}_{2n+1}} = \mathrm{i}(\bar{K}_{2n+1}[U] - K_{2n+1}[U]), \tag{21}$$

where $K_{2n+1}[U]$ is some integro-differential operator in $z, \bar{z}$. Here $t_{2n+1}, \tilde{t}_{2n+1}$ are parameters of these flows.

Statement 1. *Let $U(z, \bar{z}, t_{2n+1})$ be a solution of one of the MNV equations; let $L(t_{2n+1})$ be the corresponding two-dimensional Dirac operator* (3) *depending on an extra parameter t_{2n+1}; let $\Gamma(t_{2n+1})$ be the corresponding family of Bloch varieties.*

Then $\Gamma(t_{2n+1}) = \Gamma$ does not depend on the MNV time t_{2n+1}.

Remark 3. Here and below we use the following notational convention. If we have a complete proof of a mathematical result we call it *Theorem* or *Lemma*. If we do not have a complete strict proof yet we use the word *Statement*.

An analogous statement is well-known for soliton systems with one spatial variable. In Section 4, we prove this fact at least for algebraic-geometrical potentials, corresponding to varieties Γ of finite genus. (Sometimes such potentials are called finite-gap potentials.) It is rather clear that our proof can be extended to all smooth potentials, but to do such extension strictly we need more detailed information about analytic behavior of the Bloch functions near infinity in the momentum space, than we have now. An appropriate analytic lemma for the one-dimensional Dirac operator, corresponding to the surfaces of revolution, was proved by Schmidt in [16].

There exists a different way (may be a more natural one) to get a strict proof of Statement 1. It would be interesting to prove the following approximation property:

Conjecture 1. *Any smooth potential can be approximated by the algebraic-geometrical ones with the same periods.*

From such a result it would follow that we can restrict ourself to the algebraic–geometrical potentials in our calculations.

We have a map $U(z,\bar{z}) \to \Gamma[U]$ from the space of double-periodic real smooth functions to the space of complex subvarieties in $(\mathbb{C}\backslash 0)^2$, which is invariant under the whole MNV hierarchy. This map generates an infinite family of MNV 'normal' conservation laws. Namely, let w_1 be a generic point in $\mathbb{C}\backslash 0$. Then we have an infinite collection of numbers $w_2^{(k)}[U]$ such that $(w_1, w_2^{(k)}[U]) \in \Gamma[U]$. From Statement 1 it follows that these functionals $w_2^{(k)}[U]$ are the laws of conservation of the whole *modified Novikov–Veselov* hierarchy.

The functionals $w_2^{(k)}[U]$ are essentially nonlocal. In Section 4, we show that under the same assumptions as in Statement 1 we can expand these functionals in some asymptotic series near infinity and the expansion coefficients give us the standard 'quasi-local' conservation laws.

4. Conformal transformations of the Euclidean space $\mathbb{R}^3$, and MNV integrals of motion

In Section 3, we have associated with any double-periodic smooth real potential a Bloch variety $\Gamma[U]$. The map is constant on the trajectories of the MNV hierarchy. In this section, we associate with any immersion of a torus into $\mathbb{R}^3$ a Bloch variety Γ and show, that Γ is invariant under conformal transformations of $\mathbb{R}^3$.

Let M^2 be a torus with a fixed basis of cycles a, b. Let $X : M^2 \to \mathbb{R}^3$ be a smooth immersion of M^2 into the Euclidean space. The standard metric on $\mathbb{R}^3$ induces a conformal structure on M^2. Let z be a conformal global coordinate on the universal covering space of M^2. The coordinate z is defined uniquely up to affine transformations $z \to cz + d$. If we assume that the shift of M^2 along the a-cycle corresponds to the shift $z \to z + 1$, then the coordinate z is defined uniquely up to shifts

$$z \to z + d. \tag{22}$$

The immersion X and coordinate z define a potential $U(z,\bar{z})$ and therefore, also a Bloch variety $\Gamma[U]$. $\Gamma[U]$ is invariant under the shifts (22), thus it is completely determined by the immersion X and the cycles a, b, and we may write $\Gamma[X,a,b]$.

Conformal transformations of the Euclidean space $\mathbb{R}^3$ do not affect the conformal structure of M^2, thus they leave the coordinate z invariant up to the shifts (22). Without loss of generality we shall assume that conformal transformations of $\mathbb{R}^3$ do not change z.

Now we are in position to formulate and prove our main result:

Theorem 1. *Let $X: M^2 \to \mathbb{R}^3$ be an immersion of a torus with a fixed basis of cycles a, b into the Euclidean space; let $\Gamma[X,a,b]$, be the corresponding Bloch variety. Then $\Gamma[X,a,b]$ is invariant under conformal transformations of $\mathbb{R}^3$.*

Proof of the Theorem.

Step 1. To start with, let us recall the well-known facts about the group of conformal transformations of the standard Euclidean metric on $\mathbb{R}^3$ (or on the sphere S^3) (see for example [3]). This group is generated by the following transformations:

(1) Translations $X_i \to X_i + (X_0)_i$.
(2) Rotations $\boldsymbol{X} \to A\boldsymbol{X}$, $A \in \mathrm{SO}(3)$.
(3) Dilations $\boldsymbol{X} \to k\boldsymbol{X}$, $k \in \mathbb{R}$
(4) Inversions

$$X_i \to \frac{X_i - (X_0)_i}{\langle \boldsymbol{X} - \boldsymbol{X}_0, \boldsymbol{X} - \boldsymbol{X}_0 \rangle}. \tag{23}$$

(5) Reflections

$$\boldsymbol{X} \to \boldsymbol{X} - 2\boldsymbol{v}\langle \boldsymbol{v}, \boldsymbol{X} \rangle, \ \langle \boldsymbol{v}, \boldsymbol{v} \rangle = 1. \tag{24}$$

The connected component of the identity of this group is isomorphic to SO(1,4) (see [3, p. 143]). The corresponding Lie algebra is generated by the following basis of infinitesimal transformations:

(1) Translations P_a: $\delta X_i = \delta_{ia}$.
(2) Rotations Ω_{ab}, $a < b$: $\delta X_i = \delta_{ib} X_a - \delta_{ia} X_b$.
(3) Dilation D: $\delta X_i = X_i$
(4) Inversions K_a

$$\delta X_i = 2X_i X_a - \delta_{ia} \sum_{j=1}^{3} X_j X_j. \tag{25}$$

Step 2. Let us calculate deformations of the potential $U(z,\bar{z})$, the eigenfunction $\Psi(z,\bar{z})$ and the constants C_j in formulas (5) corresponding to all infinitesimal conformal transformations of $\mathbb{R}^3$.

(1) Translations simply shift the constants C_j and change neither $U(z,\bar{z})$ nor $\Psi(z,\bar{z})$. Thus, they do not change $\Gamma[X,a,b]$ and, without loss of generality, we can assume

$$C_j = 0, \quad j = 1,2,3. \tag{26}$$

(2) A simple direct calculation based on the representation (6) (we do not like to reproduce it here) shows that the rotations in $\mathbb{R}^3$ correspond to the following transformations of the eigenfunction $\boldsymbol{\Psi}(z,\bar{z})$

$$\begin{pmatrix} \Psi_1(z,\bar{z}) \\ \Psi_2(z,\bar{z}) \end{pmatrix} \to \alpha \begin{pmatrix} \Psi_1(z,\bar{z}) \\ \Psi_2(z,\bar{z}) \end{pmatrix} + \beta \begin{pmatrix} \bar{\Psi}_2(z,\bar{z}) \\ -\bar{\Psi}_1(z,\bar{z}) \end{pmatrix}, \quad |\alpha|^2+|\beta|^2=1, \tag{27}$$

where α and β are some complex parameters.

Both functions, $\boldsymbol{\Psi}(z,\bar{z})$ and

$$\boldsymbol{\Psi}^+(z,\bar{z}) = \begin{pmatrix} \bar{\Psi}_2(z,\bar{z}) \\ -\bar{\Psi}_1(z,\bar{z}) \end{pmatrix}, \tag{28}$$

satisfy the Dirac equation (4) with the same potential $U(z,\bar{z})$. Thus, the rotations do not change the potential and $\Gamma[X,a,b]$.

(3) The dilation is generated by the scaling transform

$$\delta\boldsymbol{\Psi}(z,\bar{z}) = \tfrac{1}{2}\boldsymbol{\Psi}(z,\bar{z}) \tag{29}$$

and changes neither $U(z,\bar{z})$ nor $\Gamma[X,a,b]$.

(4) The only nontrivial transformations of the Dirac operator correspond to the inversion generators. Up to conjugations by rotations all generators (25) are equivalent. Thus, it is sufficient to prove that $\Gamma[X,a,b]$ is invariant if we apply the generator $-K_3$:

$$\delta X_1 = -2X_1X_3, \qquad \delta K_2 = -2X_2X_3, \qquad \delta X_3 = -X_3^2 + X_1^2 + X_2^2. \tag{30}$$

Let us introduce the following notation

$$W(z,\bar{z}) = X_1(z,\bar{z}) - iX_2(z,\bar{z}). \tag{31}$$

The transformation (30) corresponds to the following transformation of the function $\boldsymbol{\Psi}(z,\bar{z})$

$$\begin{aligned} \delta\Psi_1(z,\bar{z}) &= -X_3(z,\bar{z})\Psi_1(z,\bar{z}) + \mathrm{i}W(z,\bar{z})\bar{\Psi}_2(z,\bar{z}), \\ \delta\Psi_2(z,\bar{z}) &= -X_3(z,\bar{z})\Psi_2(z,\bar{z}) - \mathrm{i}W(z,\bar{z})\bar{\Psi}_1(z,\bar{z}). \end{aligned} \tag{32}$$

Let us check it

$$\begin{aligned} \delta X_3(z,\bar{z}) = &-\int_{z_0}^{z} [(\delta\Psi_2(z',\bar{z}')\bar{\Psi}_1(z',\bar{z}') + \Psi_2(z',\bar{z}')\delta\bar{\Psi}_1(z',\bar{z}'))\,\mathrm{d}z' \\ &\quad + (\delta\Psi_1(z',\bar{z}')\bar{\Psi}_2(z',\bar{z}') + \Psi_1(z',\bar{z}')\delta\bar{\Psi}_2(z',\bar{z}'))\,\mathrm{d}\bar{z}'] \\ = &-\int_{z_0}^{z} [(-2X_3(z',\bar{z}')\Psi_2(z',\bar{z}')\bar{\Psi}_1(z',\bar{z}') \\ &\quad - \mathrm{i}W(z',\bar{z}')\bar{\Psi}_1^2(z',\bar{z}') - \mathrm{i}\bar{W}(z',\bar{z}')\Psi_2^2(z',\bar{z}'))\,\mathrm{d}z' \\ &\quad + (-2X_3(z',\bar{z}')\Psi_1(z',\bar{z}')\bar{\Psi}_2(z',\bar{z}') \\ &\quad + \mathrm{i}W(z',\bar{z}')\bar{\Psi}_2^2(z',\bar{z}') + \mathrm{i}\bar{W}(z',\bar{z}')\Psi_1^2(z',\bar{z}'))\,\mathrm{d}\bar{z}'] \end{aligned}$$

$$
\begin{aligned}
&= -\int_{z_0}^{z} [(2X_3(z',\bar z')(\partial_{z'}X_3(z',\bar z')) - W(z',\bar z')(\partial_{z'}\bar W(z',\bar z')) \\
&\qquad - \bar W(z',\bar z')(\partial_{z'}W(z',\bar z')))\,\mathrm{d}z' + (2X_3(z',\bar z')(\partial_{\bar z'}X_3(z',\bar z')) \\
&\qquad - W(z',\bar z')(\partial_{\bar z'}\bar W(z',\bar z')) - \bar W(z',\bar z')(\partial_{\bar z'}W(z',\bar z'))t)\,\mathrm{d}\bar z'] \\
&= -\int_{z_0}^{z} [(\partial_{z'}X_3^2(z',\bar z') - \partial_{z'}(W(z',\bar z')\bar W(z',\bar z')))\,\mathrm{d}z' \\
&\qquad + (\partial_{\bar z'}X_3^2(z',\bar z') - \partial_{\bar z'}(W(z',\bar z')\bar W(z',\bar z')))\,\mathrm{d}\bar z'] \\
&= W(z,\bar z)\bar W(z,\bar z) - X_3^2(z,\bar z)
\end{aligned}
$$

$$
\begin{aligned}
\delta W(z,\bar z) &= \mathrm{i}\int_{z_0}^{z} 2[\delta\Psi_2(z',\bar z')\Psi_2(z',\bar z')\,\mathrm{d}z' - \delta\Psi_1(z',\bar z')\Psi_1(z',\bar z')\,\mathrm{d}\bar z'] \\
&= \int_{z_0}^{z} 2[(-X_3(z',\bar z')i\Psi_2^2(z',\bar z') + W(z',\bar z')\bar\Psi_1(z',\bar z')\Psi_2(z',\bar z'))\,\mathrm{d}z' \\
&\qquad + (X_3(z',\bar z')i\Psi_1^2(z',\bar z') + W(z',\bar z')\bar\Psi_2(z',\bar z')\Psi_1(z',\bar z'))\,\mathrm{d}\bar z'] \\
&= 2\int_{z_0}^{z} [(-X_3(z',\bar z')(\partial_{z'}W(z',\bar z')) - W(z',\bar z')(\partial_{z'}X_3(z',\bar z')))\,\mathrm{d}z' \\
&\qquad + (-X_3(z',\bar z')(\partial_{\bar z'}W(z',\bar z')) - W(z',\bar z')(\partial_{z'}X_3(z',\bar z')))\,\mathrm{d}\bar z'] \\
&= -2\int_{z_0}^{z} [(\partial_{z'}(X_3(z',\bar z')W(z',\bar z')))\,\mathrm{d}z' \\
&\qquad + ((\partial_{\bar z'}(X_3(z',\bar z')W(z',\bar z')))\,\mathrm{d}\bar z'] \\
&= -2W(z,\bar z)X_3(z,\bar z)
\end{aligned}
$$

The corresponding transformation of the potential $U(z,\bar z)$ reads as

$$\delta U(z,\bar z) = \Psi_1(z,\bar z)\bar\Psi_1(z,\bar z) - \Psi_2(z,\bar z)\bar\Psi_2(z,\bar z). \tag{33}$$

Step 3. Let us calculate the deformation of the Bloch functions corresponding to (33).

Let $\psi(\lambda, z, \bar z)$ be the Bloch function of L. For any λ such that at least one of the functions $\operatorname{Im} p_x(\lambda)$, $\operatorname{Im} p_y(\lambda)$ is not equal to zero ('non-physical' λ) define the following pair of functions

$$
\begin{aligned}
\Omega_1(\lambda,z,\bar z) &= \int_{\infty}^{z} \psi_2(\lambda,z',\bar z')\bar\Psi_1(z',\bar z')\,\mathrm{d}z' + \psi_1(\lambda,z',\bar z')\bar\Psi_2(z',\bar z')\,\mathrm{d}\bar z', \\
\Omega_2(\lambda,z,\bar z) &= \int_{\infty}^{z} \psi_2(\lambda,z',\bar z')\Psi_2(z',\bar z')\,\mathrm{d}z' - \psi_1(\lambda,z',\bar z')\Psi_1(z',\bar z')\,\mathrm{d}\bar z',
\end{aligned} \tag{34}
$$

where the integrals are taken along an arbitrary path in the z-plane, connecting the points z and ∞ such that the integrand decays exponentially along this path. The integrands in (34) are closed 1-forms, thus the integrals do not depend on a concrete choice of the path. Using the same arguments as in Section A.2 we may easily prove that $\Omega_1(\lambda, z, \bar{z})$ and $\Omega_2(\lambda, z, \bar{z})$ are meromorphic in λ outside infinity.

The function $\Psi(z, \bar{z})$ is double-periodic or anti-periodic in z (see (9)), thus the functions $\Omega_1(\lambda, z, \bar{z})$, $\Omega_2(\lambda, z, \bar{z})$ have the following periodicity properties:

$$\begin{aligned} \Omega_k(\lambda, z+1, \bar{z}+1) &= \mathcal{W}_1 w_1(\lambda)\Omega_k(\lambda, z, \bar{z}) \\ \Omega_k(\lambda, z+\tau, \bar{z}+\bar{\tau}) &= \mathcal{W}_2 w_2(\lambda)\Omega_k(\lambda, z, \bar{z}) \end{aligned} \quad k = 1, 2. \tag{35}$$

(see (9) for the definition of $\mathcal{W}_1$, $\mathcal{W}_2$.)

Lemma 1. *The variation of the Bloch function $\psi(\lambda, z, \bar{z})$ corresponding to* (33) *reads as*

$$\begin{aligned} \delta\psi_1(\lambda, z, \bar{z}) &= \Omega_1(\lambda, z, \bar{z})\Psi_1(z, \bar{z}) - \Omega_2(\lambda, z, \bar{z})\bar{\Psi}_2(z, \bar{z}) + \alpha(\lambda)\psi_1(\lambda, z, \bar{z}), \\ \delta\psi_2(\lambda, z, \bar{z}) &= \Omega_1(\lambda, z, \bar{z})\Psi_2(z, \bar{z}) + \Omega_2(\lambda, z, \bar{z})\bar{\Psi}_1(z, \bar{z}) + \alpha(\lambda)\psi_2(\lambda, z, \bar{z}), \end{aligned} \tag{36}$$

where $\alpha(\lambda)$ is some meromorphic function, fixed by the normalization condition:

$$\delta\psi_1(\lambda, z, \bar{z}) + \delta\psi_2(\lambda, z, \bar{z})|_{z=z_1} = 0. \tag{37}$$

Proof of Lemma 1. To start with, let us recall a simple fact from Bloch theory (see for example [10]).

Generically, if we calculate variations of the Bloch function, we deform Γ, and we cannot assume both $\delta w_1(\lambda) = 0$ and $\delta w_2(\lambda) = 0$ simultaneously. To compare functions on different subvarieties in $\mathbb{C}^2$, we have to fix some connection. The simplest way to do this is to assume $\delta w_1(\lambda) = 0$.

Then the variation of the Bloch function can be found as the unique solution of the linearized Dirac equation

$$\delta L\psi(\lambda, z, \bar{z}) + L\delta\psi(\lambda, z, \bar{z}) = 0 \tag{38}$$

satisfying (37) such that

$$\delta\psi(\lambda, t_1 + \tau t_2, t_1 + \bar{\tau} t_2) = O([1 + |t_2|]e^{ip_1(\lambda)t_1 + ip_2(\lambda)t_2}). \tag{39}$$

A simple direct calculation shows that (36) solves (38). From (35) and (9) it follows, that

$$\begin{aligned} \delta\psi(w_1, w_2, z+1, \bar{z}+1) &= w_1\delta\psi(w_1, w_2, z, \bar{z}), \\ \delta\psi(w_1, w_2, z+\tau, \bar{z}+\bar{\tau}) &= w_2\delta\psi(w_1, w_2, z, \bar{z}), \end{aligned} \tag{40}$$

thus variations of the type (36) satisfy (39). This completes the proof. □

The function $\delta\psi(\lambda, z, \bar{z})$ defined by (36) has the same periodicity properties as the original Bloch function $\psi(\lambda, z, \bar{z})$ (see formulas (11) and (40) respectively). Thus, our special variations satisfy $\delta w_1(\lambda) = 0$ and $\delta w_2(\lambda) = 0$ simultaneously.

Step 4. From (36) it follows that if we apply infinitesimal conformal transformation $-K_3$ to our immersion, the Bloch functions of the deformed Dirac operator are meromorphic on the same variety $\Gamma[X, a, b]$ as the original Bloch functions and have the same multipliers w_1, w_2. Thus, our deformation does not change $\Gamma[X, a, b]$. This completes the proof of Theorem 1. □

Example 1 Surfaces of revolution. Let γ be a closed non-self-intersecting curve in the half-plane $X_2 = 0$, $X_1 > 0$ in $\mathbb{R}^3$. Rotating γ about the axes $X_1 = X_2 = 0$ we get a surface of revolution. It is always a torus with a fixed pair of periods.

Such surfaces are essentially simpler from the soliton point of view. Potentials $U(z, \bar{z})$ corresponding to such surfaces depend only on one real variable $x = \operatorname{Re} z$. Instead of the fixed energy spectral transform for the two-dimensional Dirac operator, we have the spectral transform for the 2×2 first-order matrix differential operator in one variable. Periodic direct spectral transform for such operators (for both finite-gap and infinite-gap potentials) was developed by Schmidt [16].

The MNV equations for surfaces of revolution are reduced to the well-studied modified Korteweg-de Vries equations (MKdV). In contrast with MNV, all higher MKdV conservation laws are local in terms of the potential. Eq. (33) in this situation was integrated by Melnikov [14] in the class of potentials sufficiently fast decaying at infinity. Our theorem for the surfaces of revolution does not follow formally from [14] because the periodic MKdV theory and the decay at infinity use different technical tools. Nevertheless, it is possible to essentially simplify our proof in this specific case.

Appendix A. Modified Novikov–Veselov equations with periodic boundary conditions

In this appendix, we discuss the zero-energy spectral transform for the double-periodic Dirac operator (and the generalized Weierstrass transform) from the soliton point of view. This transform is naturally connected with a completely integrable hierarchy of integro-differential equations with two spatial variables known as modified Novikov–Veselov hierarchy (MNV).

Formally, the results of this section are not used in our proof of Theorem 1, but we hope they allow the reader to gain a better understanding of the problem.

There is a rather large number of papers dedicated to the periodic problem for soliton equations (see, e.g. the textbook [21]). Nevertheless, the direct spectral transform for two-dimensional Dirac operator was never studied in such context in the literature available to us. Important properties of Bloch varieties for multidimensional Dirac operators were proved in [12], but they are not sufficient for the purpose of integrating the periodic MNV equations.

From the point of view of the Bloch theory, the two-dimensional double-periodic Dirac operator is rather similar to the two-dimensional double-periodic Schrödinger operator. The fixed-energy direct spectral transform for the latter was constructed Krichever [10] using soliton methods. This problem was studied in more detail by Feldman et al. (see [4]). In

Section A.1, we describe the structure of the Dirac–Bloch variety by methods analogous to [10]. It is important to remark that, in spite of the similarity between these two problems, we have to overcome some additional technical difficulties on the way. The asymptotic expansion of the Bloch variety gives us 'quasi-local' MNV conservation laws.

The zero-energy scattering problem for the two-dimensional Dirac operator possesses an infinite-dimensional algebra of symmetries, generated by the MNV equations. These were constructed by Bogdanov [2]. In [2], a generalization of the Miura transform was defined, and it was shown that this transform maps the MNV equations to the Novikov–Veselov hierarchy (see [19]), associated with the fixed-energy two-dimensional Schrödinger operator. MNV equations in the space of functions, decaying at infinity, were integrated by the so-called method of $\bar\partial$-problem in [2]. Periodic MNV theory is discussed in Section A.3.

In contrast with the one-dimensional soliton systems, the two-dimensional ones essentially depend on the boundary conditions. To define the periodic MNV hierarchy uniquely, we have to fix some constants of integration. One of the simplest way to do it is to define the MNV hierarchy in terms of the so-called Cauchy–Baker–Akhiezer (CBA) kernel. This kernel was introduced for the Kadomtsev–Petviashvily hierarchy by Orlov and Grinevich [6] (see also [7]). The MNV hierarchy in terms of the CBA kernel is discussed in Section A.2.

Here, we always assume that $U(z,\bar z)$ is real, smooth and double-periodic (8).

A.1. Asymptotic structure of the Bloch variety

For large $\operatorname{Im} p_1$ and $\operatorname{Im} p_2$, the structure of the Bloch variety can be studied by the perturbation theory. Following [10], we start from the Dirac operator with zero potential $U(z,\bar z)\equiv 0$. The corresponding Bloch variety is a union of two Riemann spheres $\Gamma^{(0)} = \Gamma_1^{(0)}\cup\Gamma_2^{(0)}$, $\Gamma_1^{(0)}=\Gamma_2^{(0)}=\mathbb{CP}^1$ with a coordinate λ and

$$\psi(\lambda,z,\bar z)=\begin{pmatrix}1\\0\end{pmatrix}e^{\lambda\bar z},\quad \lambda\in\Gamma_1^{(0)},\qquad \psi(\lambda,z,\bar z)=\begin{pmatrix}0\\1\end{pmatrix}e^{\lambda z},\quad \lambda\in\Gamma_2^{(0)}. \tag{A.1}$$

A pair $\lambda_1\in\Gamma_1^{(0)}$, $\lambda_2\in\Gamma_2^{(0)}$ is called resonant if

$$e^{\lambda_1-\lambda_2}=1,\qquad e^{\lambda_1\bar\tau-\lambda_2\tau}=1, \tag{A.2}$$

and non-resonant otherwise. All resonant pairs are given by the following formulas:

$$\lambda_1^{(m,n)}=\frac{\pi m\operatorname{Re}\tau-\pi n}{\operatorname{Im}\tau}+i\pi m,\qquad \lambda_2^{(m,n)}=\bar\lambda_1^{(m,n)},\quad m,n\in\mathbb{Z}. \tag{A.3}$$

Let us call a point $\lambda\in\Gamma_1^{(0)}$ non-resonant, if the pair $\lambda\in\Gamma_1^{(0)}$ and $\bar\lambda\in\Gamma_2^{(0)}$ is non-resonant. The antiholomorphic involution $\sigma\theta$ maps $\Gamma_2^{(0)}$ to $\Gamma_1^{(0)}$, thus it is sufficient to develop a perturbation theory only on $\Gamma_1^{(0)}$.

Let ε, R be some positive constants. Denote by $\Gamma_{\varepsilon,R}^{(0)}$ the domain obtained from $\mathbb{CP}^1$ by removing ε neighbourhoods of all resonant points $\lambda_1^{(m,n)}$ and the disk $|\lambda|\le R$.

Lemma 2. *For any $\varepsilon > 0$, there exists a constant $R(\varepsilon)$ such that in the domain $\Gamma^{(0)}_{\varepsilon,R(\varepsilon)}$ there exists an unique solution of the Dirac equation* (4) *with normalization* (12) *such that*

$$\begin{aligned} \psi(\lambda, z+1, \bar{z}+1) &= \mathrm{e}^{\lambda+h(\lambda)}\psi(\lambda, z, \bar{z}), \\ \psi(\lambda, z+\tau, \bar{z}+\bar{\tau}) &= \mathrm{e}^{\lambda\bar{\tau}+h(\lambda)\tau}\psi(\lambda, z, \bar{z}), \end{aligned} \tag{A.4}$$

where $h(\lambda)$ is uniquely defined under the condition that $h(\lambda) \to 0$ as $\lambda \to \infty$. The functions $\psi(\lambda, z, \bar{z})$, and $h(\lambda)$ are holomorphic in λ in the domain $\Gamma^{(0)}_{\varepsilon,R(\varepsilon)}$.

A proof of this statement is to appear in a forthcomming paper by Taimanov and Schmidt. It is rather long and quite technical. We do not want to present it here.

Statement 2. *The functions $\psi(\lambda, z, \bar{z})$ and $h(\lambda)$, defined in Lemma 2, possess the following asymptotic expansions as $\lambda \to \infty$*

$$\psi(\lambda, z, \bar{z}) = \mathrm{e}^{\lambda(\bar{z}-\bar{z}_1)+h(\lambda)(z-z_1)} \begin{pmatrix} 1 + \dfrac{\phi_1(z,\bar{z})}{\lambda} + \dfrac{\phi_2(z,\bar{z})}{\lambda^2} + \dfrac{\phi_3(z,\bar{z})}{\lambda^3} + \dfrac{\phi_4(z,\bar{z})}{\lambda^4} + \cdots \\ \dfrac{\chi_1(z,\bar{z})}{\lambda} + \dfrac{\chi_2(z,\bar{z})}{\lambda^2} + \dfrac{\chi_3(z,\bar{z})}{\lambda^3} + \dfrac{\chi_4(z,\bar{z})}{\lambda^4} + \cdots \end{pmatrix} \tag{A.5}$$

$$h(\lambda) = \frac{h_1}{\lambda} + \frac{h_3}{\lambda^3} + \frac{h_5}{\lambda^5} + \cdots \tag{A.6}$$

(The Bloch variety Γ has a symmetry $\sigma : (w_1, w_2) \to (w_1^{-1}, w_2^{-1})$, $\sigma(\lambda) = -\lambda$, thus all even coefficients in (A.6) are identically zero.)

Unfortunately, at this moment we do not have any complete proof of the forgoing Statement. It is rather clear how to do it, but this proof needs a somewhat lengthy asymptotic analysis, and we are not in a position to do it presently. But we know that it is fulfilled at least in two important specific situations:

(1) $U(z, \bar{z})$ is an algebraic–geometrical (or, equivalently, finite-gap) potential. It means, that the normalization of the zero-energy Bloch variety is algebraic (has finite genus).
(2) $U(z, \bar{z})$ depend only on one real variable, $x = \operatorname{Re} z$. This fact was proved by Schmidt [16]. Such potentials corresponds to surfaces of revolution.

We would like to remark that, in [4], a class of Riemann surfaces was introduced, which are in some sense similar to compact Riemann surfaces, and the zero-energy level of a two-dimensional Schrödinger operator belongs to this class. Therefore, it is natural to expect that the zero-energy level of the two-dimensional Dirac operator also belongs to this class.

To define the modified Novikov–Veselov equations (MNV) and their laws of conservation, it is sufficient to have a formal solution of the Dirac equation in the form (A.5) and (A.6). Let us show that such a solution exists and is unique, if we assume that all $\phi_k(z, \bar{z})$, $\chi_k(z, \bar{z})$ are bounded in the whole z-plane.

Inserting (A.5) and

$$h(\lambda) = \frac{h_1}{\lambda} + \frac{h_2}{\lambda^2} + \frac{h_3}{\lambda^3} + \cdots \tag{A.7}$$

in (4), we get the following system of equations

$$\begin{aligned} &\chi_1(z,\bar z) = -U(z,\bar z),\\ &\chi_k(z,\bar z) = -\partial_{\bar z}\chi_{k-1}(z,\bar z) - U(z,\bar z)\phi_{k-1}(z,\bar z), \quad k > 1,\\ &\partial_z\phi_k(z,\bar z) = U(z,\bar z)\chi_k(z,\bar z) - h_k - \sum_{j=1}^{k-1} h_j\phi_{k-j}(z,\bar z). \end{aligned} \tag{A.8}$$

We solve this system by induction. First, we find $\chi_1(z,\bar z)$; then $\phi_1(z,\bar z)$; then $\chi_2(z,\bar z)$; then $\phi_2(z,\bar z)$ and so on. To find $\chi_k(z,\bar z)$, at each step we differentiate some double-periodic functions, obtained at previous steps. Thus, they are defined uniquely and are automatically double-periodic. To find $\phi_k(z,\bar z)$, we have to invert the operator ∂_z in the space of functions bounded in the whole z-plane (any bounded solution $\phi_k(z,\bar z)$ is automatically double-periodic). This is possible if, and only if, the mean value of the right-hand side is equal to zero:

$$\left\langle\left\langle U(z,\bar z)\chi_k(z,\bar z) - h_k - \sum_{j=1}^{k-1} h_j\phi_{k-j}(z,\bar z)\right\rangle\right\rangle = 0, \tag{A.9}$$

where

$$\langle\langle F(z,\bar z)\rangle\rangle = \int_0^1\int_0^1 \mathrm{d}t_1\,\mathrm{d}t_2 F(t_1+\tau t_2, t_1+\bar\tau t_2). \tag{A.10}$$

Thus, at each step we find h_k from (A.9) and then calculate $\phi_k(z,\bar z)$. The function $\phi_k(z,\bar z)$ is determined by the system (A.8) uniquely up to adding an arbitrary constant. This constant is fixed by the normalization condition (12).

Let us check that the function $h(\lambda)$ does not depend on the normalization point z_1. If we change the point z_1, then we change the integration constants. But these constants can be arbitrarily shifted by multiplying the whole solution to a formal series in λ

$$\begin{aligned} &\begin{bmatrix} 1 + \dfrac{\phi_1(z,\bar z)}{\lambda} + \dfrac{\phi_2(z,\bar z)}{\lambda^2} + \cdots \\ \dfrac{\chi_1(z,\bar z)}{\lambda} + \dfrac{\chi_2(z,\bar z)}{\lambda^2} + \cdots \end{bmatrix} \to \left[1 + \frac{\alpha_1}{\lambda} + \frac{\alpha_2}{\lambda^2} + \cdots\right]\\ &\times \begin{bmatrix} 1 + \dfrac{\phi_1(z,\bar z)}{\lambda} + \dfrac{\phi_2(z,\bar z)}{\lambda^2} + \cdots \\ \dfrac{\chi_1(z,\bar z)}{\lambda} + \dfrac{\chi_2(z,\bar z)}{\lambda^2} + \cdots \end{bmatrix}. \end{aligned} \tag{A.11}$$

This multiplication does not affect $h(\lambda)$.

We have proved that the constants h_1, h_3, h_5, ... are completely determined by the potential $U(z, \bar{z})$ and all $h_{2k} = 0$. Thus, we have constructed an infinite sequence of functionals $h_{2k+1}[U]$. The formulas for the first two of them are:

$$h_1 = -\langle\langle U^2(z, \bar{z})\rangle\rangle, \tag{A.12}$$
$$h_3 = -\langle\langle U(z, \bar{z})U_{\bar{z}\bar{z}}(z, \bar{z}) - (U^2(z, \bar{z}) + h_1)V_{1\bar{z}}(z, \bar{z})\rangle\rangle, \tag{A.13}$$

where

$$V_1(z, \bar{z}) = \partial_z^{-1}(U^2(z, \bar{z}) + h_1). \tag{A.14}$$

Let us point out that adding an arbitrary constant to $V_1(z, \bar{z})$ does not affect h_3.

We have defined an infinite collection of functionals $h_{2k+1}[U]$, $k = 0, 1, 2, \ldots$ The following statement explains why these functionals are so important.

Statement 3. *The quantities $h_{2k+1}[U]$ are laws of conservation for the whole hierarchy of the modified Novikov–Veselov equations (MNV).*

Assuming that Statement 2 is fulfilled, we will prove this Statement at the end of the section.

It is well-known in the soliton theory that integrable systems with one spatial variable, usually have infinitely many local laws of conservation . For multidimensional soliton systems, we normally have the opposite situation: almost all laws of conservation are non-local. Let us briefly discuss the case of the MNV hierarchy.

A functional $Q[U]$ is called local if it possesses the following representation:

$$Q[U] = \langle\langle q(U, U_z, U_{\bar{z}}, U_{zz}, U_{z\bar{z}}, U_{\bar{z}\bar{z}}, \ldots)\rangle\rangle, \tag{A.15}$$

where the density $q(\ldots)$ depend only on $U(z, \bar{z})$ and a finite number of its derivatives. Of course $h_1[U]$ is local. The next laws of conservation $h_3[U]$ is non-local because the corresponding density depends on an auxiliary function $V_{1\bar{z}}(z, \bar{z})$, and to calculate $V_{1\bar{z}}(z, \bar{z})$ we have to know $U(z, \bar{z})$ on the entire z-plane. It it rather evident that higher functionals, $h_{2k+1}[U]$, are also non-local. This non-locality creates no serious problems if the potential is double-periodic, but it is very difficult to extend existing definitions to wider classes of boundary conditions.

In [10], the perturbation theory was developed also in the neighbourhood of resonant pairs. It can be shown that, for sufficiently large λ, the surface Γ is obtained from $\Gamma(0)$ by attaching small handles to the resonant pairs. We do not use this fact; thus, we do not want to discuss it now. However, we use the following property of Γ:

Corollary 1. *The Bloch variety Γ has two infinite points corresponding to the points $\lambda = \infty$ in $\Gamma_1^{(0)}$ and $\Gamma_2^{(0)}$. We denote them by ∞_+ and ∞_-, respectively.*

This statement follows immediately from Lemma 2.

Further, we shall use the following notation: $\lambda \to \infty_+$, where λ is a point of Γ, always means that λ tends to ∞_+ in the domain $\Gamma^{(0)}_{\varepsilon,R(\varepsilon)}$; the notation $\lambda \to \infty_-$ always means that $\bar{\lambda} \in \Gamma^{(0)}_{\varepsilon,R(\varepsilon)}$.

A.2. Cauchy–Baker–Akhiezer kernel

To study symmetries of the soliton equation, it is convenient to use the Cauchy–Baker–Akhiezer kernel (CBA) (see [6]). In this section, we define the CBA kernel on the zero-energy Bloch variety of the two-dimensional Dirac operator.

To start with, suppose (λ, μ) is a pair of points in Γ such that

(1) $\psi(\nu, z, \bar{z})$ is non-singular at the points $\nu = \lambda$ and $\nu = \sigma\mu$. (Let us recall that, outside $\nu = \infty$, the poles of $\psi(\nu, z, \bar{z})$ arose due to normalization (12) and do not depend on z and $\bar{z}$.)

(2) At least one of the following conditions is fulfilled:

$$\operatorname{Im} p_1(\lambda) - \operatorname{Im} p_1(\mu) \neq 0 \text{ or } \operatorname{Im} p_2(\lambda) - \operatorname{Im} p_2(\mu) \neq 0. \tag{A.16}$$

Then, we may define $\tilde{\omega}(\lambda, \mu, z, \bar{z})$ by:

$$\tilde{\omega}(\lambda, \mu, z, \bar{z}) = \int_{\infty}^{z} \mathrm{d}\tilde{\omega}(\lambda, \mu, z', \bar{z}'), \tag{A.17}$$

where

$$\mathrm{d}\tilde{\omega}(\lambda, \mu, z', \bar{z}') = \psi_2(\lambda, z', \bar{z}')\psi_2(\sigma\mu, z', \bar{z}')\,\mathrm{d}z' - \psi_1(\lambda, z', \bar{z}')\psi_1(\sigma\mu, z', \bar{z}')\,\mathrm{d}\bar{z}'. \tag{A.18}$$

The integral in (A.17) is taken along some path γ in the z-plane such that

(1) γ connects the point z with ∞.

(2) The form $\mathrm{d}\tilde{\omega}(\lambda, \mu, z', \bar{z}')$ decays exponentially as $z' \to \infty$ along γ.

Condition (A.16) guarantees the existence of such a path.

From the Dirac equation (4), if follows that the form $\mathrm{d}\tilde{\omega}(\lambda, \mu, z', \bar{z}')$ is closed and that the integral (A.17) is well-defined.

The next step is to show that, for a fixed μ, our function $\tilde{\omega}(\lambda, \mu, z, \bar{z})$ is meromorphic in λ on Γ outside the points ∞_+ and ∞_-.

Suppose that a pair (λ, μ) satisfies the following condition, which is much weaker than (A.16).

(2′). At least one of the following combinations:

$$\frac{1}{2\pi}(p_1(\lambda) - p_1(\mu)) \text{ or } \frac{1}{2\pi|\tau|}(p_2(\lambda) - p_2(\mu)) \text{ is non-integer} \tag{A.19}$$

or, equivalently, at least one of the following inequalities is fulfilled

$$w_1(\lambda)w_1^{-1}(\mu) \neq 1, \text{ or } w_2(\lambda)w_2^{-1}(\mu) \neq 1. \tag{A.20}$$

Assume, that $w_1(\lambda)w_1^{-1}(\mu) \neq 1$. Then, we have either

$$\tilde{\omega}(\lambda, \mu, z, \bar{z}) = \int_{-\infty}^{0} \tilde{\omega}_x(\lambda, \mu, z+t, \bar{z}+t)\,\mathrm{d}t \tag{A.21}$$

or

$$\tilde{\omega}(\lambda, \mu, z, \bar{z}) = -\int_{0}^{\infty} \tilde{\omega}_x(\lambda, \mu, z+t, \bar{z}+t)\,\mathrm{d}t, \tag{A.22}$$

where

$$\tilde{\omega}_x(\lambda, \mu, z', \bar{z}') = \psi_2(\lambda, z', \bar{z}')\psi_2(\sigma\mu, z', \bar{z}') - \psi_1(\lambda, z', \bar{z}')\psi_1(\sigma\mu, z', \bar{z}') \tag{A.23}$$

and $t \in \mathbb{R}$.

If $|w_1(\lambda)w_1^{-1}(\mu)| \geq 1$, from (A.21) we get:

$$\begin{aligned}\tilde{\omega}(\lambda, \mu, z, \bar{z}) &= \left(\int_{-1}^{0} + \int_{-2}^{-1} + \int_{-3}^{-2} + \cdots\right) \tilde{\omega}_x(\lambda, \mu, z+t, \bar{z}+t)\,\mathrm{d}t \\ &= \left(\frac{w_1(\mu)}{w_1(\lambda)} + \frac{w_1^2(\mu)}{w_1^2(\lambda)} + \frac{w_1^3(\mu)}{w_1^3(\lambda)} + \cdots\right) \int_0^1 \tilde{\omega}_x(\lambda, \mu, z+t, \bar{z}+t)\,\mathrm{d}t \\ &= \frac{w_1(\mu)}{w_1(\lambda) - w_1(\mu)} \int_0^1 \tilde{\omega}_x(\lambda, \mu, z+t, \bar{z}+t)\,\mathrm{d}t.\end{aligned} \tag{A.24}$$

Similarly, if $|w_1(\lambda)w_1^{-1}(\mu)| \leq 1$, from (A.22) we get:

$$\tilde{\omega}(\lambda, \mu, z, \bar{z}) = \frac{w_1(\mu)}{w_1(\lambda) - w_1(\mu)} \int_0^1 \tilde{\omega}_x(\lambda, \mu, z+t, \bar{z}+t)\,\mathrm{d}t. \tag{A.25}$$

The formulas (A.24) and (A.25) define meromorphic continuations of the function $\tilde{\omega}(\lambda, \mu, z, \bar{z})$ to the entire surface Γ. To conclude the proof, it remains to note that formulas (A.24) and (A.25) coincide.

If $w_2(\lambda)w_2^{-1}(\mu) \neq 1$, then the integration path can be chosen along the line $z' = z + \tau t$, and we obtain a new representation for the same function $\tilde{\omega}(\lambda, \mu, z, \bar{z})$:

$$\tilde{\omega}(\lambda, \mu, z, \bar{z}) = \frac{w_2(\mu)}{w_2(\lambda) - w_2(\mu)} \int_0^1 \tilde{\omega}_2(\lambda, \mu, z+\tau t, \bar{z}+\bar{\tau} t)\,\mathrm{d}t, \tag{A.26}$$

where

$$\tilde{\omega}_2(\lambda, \mu, z', \bar{z}') = \tau \psi_2(\lambda, z', \bar{z}')\psi_2(\sigma\mu, z', \bar{z}') - \bar{\tau}\psi_1(\lambda, z', \bar{z}')\psi_1(\sigma\mu, z', \bar{z}'). \tag{A.27}$$

Now we are in position to define the Cauchy–Baker–Akhiezer kernel $\omega(\lambda, \mu, z, \bar{z})$:

$$\omega(\lambda, \mu, z, \bar{z}) = -\frac{1}{2\pi} \frac{\mathrm{d}p_1(\mu)}{\langle \tilde{\omega}_x(\mu, \mu, z, \bar{z})\rangle_x} \tilde{\omega}(\lambda, \mu, z, \bar{z}), \tag{A.28}$$

where

$$\langle \tilde{\omega}_x(\mu, \mu, z, \bar{z})\rangle_x = \int_0^1 \tilde{\omega}_x(\lambda, \mu, z+t, \bar{z}+t)\,\mathrm{d}t. \tag{A.29}$$

It is a simple exercise to check that the function $\langle \tilde{\omega}_x(\mu, \mu, z, \bar{z})\rangle_x$ does not depend on z and is even in μ, i.e.

$$\langle \tilde{\omega}_x(\mu, \mu, z, \bar{z})\rangle_x = \langle \tilde{\omega}_x(\sigma\mu, \sigma\mu, z, \bar{z})\rangle_x, \tag{A.30}$$

$$\langle \tilde{\omega}_x(\mu, \mu, z, \bar{z})\rangle_x = \mp 1 + \frac{c_2^{(\pm)}}{\mu^2} + \frac{c_4^{(\pm)}}{\mu^4} + \cdots, \quad \mu \to \infty_{\pm}. \tag{A.31}$$

Lemma 3. *The Cauchy–Baker–Akhiezer kernel $\omega(\lambda, \mu, z, \bar{z})$, defined here, has the following properties:*

(1) *For any fixed z, $\omega(\lambda, \mu, z, \bar{z})$ is a meromorphic function of λ and a meromorphic 1-form in μ (both on the finite part of Γ).*

(2) *For generic μ, $\omega(\lambda, \mu, z, \bar{z})$ has poles at the poles of $\psi(\lambda, z, \bar{z})$ and at the point $\lambda = \mu$. It is holomorphic outside these points on the finite part of Γ.*

(3) *For a generic λ and $\mu \to \lambda$*

$$\omega(\lambda, \mu, z, \bar{z}) = \frac{1}{2\pi \mathrm{i}} \frac{\mathrm{d}\mu}{\mu - \lambda} + \textit{regular terms}, \tag{A.32}$$

or, equivalently,

$$\oint_{\mu \in S} \omega(\lambda, \mu, z, \bar{z}) = 1, \tag{A.33}$$

where S is a small contour surrounding the point λ.

(4) *Let $\mu \to \infty_+$ and let λ be a fixed point in Γ. Then, we have the following formal expansion:*

$$\omega(\lambda, \mu, z, \bar{z}) = \frac{\mathrm{d}\mu}{2\pi \mathrm{i}} \left[\frac{\psi_1(\lambda, z, \bar{z})}{\mu} + \sum_{k=2}^{\infty} \frac{R_k^{(+)}[\partial_{\bar{z}}]\psi_1(\lambda, z, \bar{z})}{\mu^k} \right] \mathrm{e}^{-\mu\bar{z} - h(\mu) z}, \tag{A.34}$$

where

$$R_k^{(+)}[\partial_{\bar z}] = \partial_{\bar z}^{k-1} + \sum_{l=0}^{k-2} v_{kl}^{(+)}(z,\bar z)\partial_{\bar z}^{l} \tag{A.35}$$

are differential operators in $\bar z$ *of order* $k-1$. *The coefficients* $v_{kl}^{(+)}(z,\bar z)$ *do not depend on* λ *and* μ; *they are differential polynomials of the asymptotic expansion coefficients* $\phi_j(z,\bar z)$, $\chi_j(z,\bar z)$, $j<k$. *Also, each function* $v_{kl}^{(+)}(z,\bar z)$ *depends on a finite number of constants* h_{2j+1}, $c_{2j}^{(\pm)}$.

Similarly, for $\mu\to\infty_-$, *we have*

$$\omega(\lambda,\mu,z,\bar z) = \frac{\mathrm{d}\mu}{2\pi\mathrm{i}}\left[\frac{\psi_2(\lambda,z,\bar z)}{\mu} + \sum_{k=2}^{\infty}\frac{R_k^{(-)}[\partial_z]\psi_2(\lambda,z,\bar z)}{\mu^k}\right]\mathrm{e}^{-\mu z-h(\mu)\bar z}, \tag{A.36}$$

where

$$R_k^{(-)}[\partial_z] = \partial_z^{k-1} + \sum_{l=0}^{k-2} v_{kl}^{(-)}(z,\bar z)\partial_z^{l}. \tag{A.37}$$

(5)
$$\begin{aligned}\omega(\lambda,\mu,z+1,\bar z+1) &= w_1(\lambda)w_1^{-1}(\mu)\omega(\lambda,\mu,z,\bar z),\\ \omega(\lambda,\mu,z+\tau,\bar z+\bar\tau) &= w_2(\lambda)w_2^{-1}(\mu)\omega(\lambda,\mu,z,\bar z).\end{aligned} \tag{A.38}$$

Statement 4. *If Statement* 2 *is fulfilled, then the CBA kernel has the following additional properties:*

(1) *The formal expansions* (*A*.34), (*A*.36) *are asymptotic.*

(2) *Let* $f^{(+)}(\lambda,z,\bar z)$ *be the following formal series in* λ

$$f^{(+)}(\lambda,z,\bar z) = \left\{\sum_{k=-N}^{\infty}\frac{f_k^{(+)}(z,\bar z)}{\lambda^k}\right\}\mathrm{e}^{\lambda\bar z}. \tag{A.39}$$

Then

$$2\pi\mathrm{i}\,\mathrm{res}|_{\mu=\infty_+}\omega(\lambda,\mu,z,\bar z)f^{(+)}(\mu,z,\bar z) = \left\{\sum_{k=-N}^{0}\frac{f_k^{(+)}(z,\bar z)}{\lambda^k} + \mathrm{O}\left(\frac{1}{\lambda}\right)\right\}\mathrm{e}^{\lambda\bar z} \tag{A.40}$$

as $\lambda\to\infty_+$ *and*

$$2\pi\mathrm{i}\,\mathrm{res}|_{\mu=\infty_+}\omega(\lambda,\mu,z,\bar z)f^{(+)}(\mu,z,\bar z) = \mathrm{O}\left(\frac{1}{\lambda}\right)\mathrm{e}^{\lambda z} \tag{A.41}$$

as $\lambda\to\infty_-$.

Similarly, if $f^{(-)}(\lambda,z,\bar z)$, *is the following formal series in* λ

$$f^{(-)}(\lambda,z,\bar z) = \left\{\sum_{k=-N}^{\infty}\frac{f_k^{(-)}(z,\bar z)}{\lambda^k}\right\}\mathrm{e}^{\lambda z} \tag{A.42}$$

then

$$2\pi \mathrm{i}\, \mathrm{res}|_{\mu=\infty_-} \omega(\lambda, \mu, z, \bar{z}) f^{(-)}(\mu, z, \bar{z}) = \left\{ \sum_{k=-N}^{0} \frac{f_k^{(-)}(z, \bar{z})}{\lambda^k} + \mathrm{O}\left(\frac{1}{\lambda}\right) \right\} \mathrm{e}^{\lambda z} \tag{A.43}$$

as $\lambda \to \infty_-$, *and*

$$2\pi \mathrm{i}\, \mathrm{res}|_{\mu=\infty_-} \omega(\lambda, \mu, z, \bar{z}) f^{(-)}(\mu, z, \bar{z}) = \mathrm{O}\left(\frac{1}{\lambda}\right) \mathrm{e}^{\lambda \bar{z}} \tag{A.44}$$

as $\lambda \to \infty_+$.

Remark 4. The formulas (A.40)–(A.44) require some comments. On the left-hand side, we have the formal series in μ, thus the analytic definition of the residue as an integral does not work. Fortunately the terms, containing exponents in μ, annihilate each other and we can define the residue as the coefficient of $1/\mu$:

$$\mathrm{res}|_{\mu=\infty} \sum_{k=-N}^{\infty} \frac{a_k}{\mu^k}\, \mathrm{d}\mu = a_1. \tag{A.45}$$

Proof of Lemma 3.

(1) We have constructed $\tilde{\omega}(\lambda, \mu, z, \bar{z})$ as a meromorphic function. The function $\langle \tilde{\omega}_x(\mu, \mu, z, \bar{z}) \rangle_x$ is meromorphic in μ on the entire Γ and $\mathrm{d}p_1(\mu)$ is a meromorphic 1-form on Γ. Thus, (A.28) gives us a meromorphic function with appropriate tensor properties.

(2) From Lemma 2, it follows that, for generic μ,
   (a) $\psi(\nu, z, \bar{z})$ is non-singular at the points $\nu = \mu$, $\nu = \sigma\mu$,
   (b) $\langle \omega_x(\mu, \mu, z, \bar{z}) \rangle_x \neq 0$
   (c) $w_1(\lambda) w_1^{-1}(\mu) = w_2(\lambda) w_2^{-1}(\mu) = 1, \quad \text{iff } \lambda = \mu$.

   If μ fulfill these conditions, if $\lambda \neq \mu$ and, if $\psi(\lambda, z, \bar{z})$ is non-singular, then formulas (A.25)–(A.28) define a non-singular function of z, $\bar{z}$.

(3) Let $\lambda \to \mu$. To calculate the asymptotes of the CBA kernel, we expand the function $w_1(\lambda)$ in (A.24) in a series in $\lambda - \mu$, using (13) and (14). We see, that we have a first-order pole and the normalization in (A.28) is chosen so that the residue is exactly 1.

(4) To calculate the asymptotes of $\tilde{\omega}(\lambda, \mu, z, \bar{z})$, as $\mu \to \infty$, we substitute the asymptotic expansion (A.5) to (A.17) and use the following asymptotic formula:

$$\int^{z} \mathrm{e}^{\lambda \bar{z}} f(\lambda, z', \bar{z}')\, \mathrm{d}z' + \mathrm{e}^{\lambda \bar{z}} g(\lambda, z', \bar{z}')\, \mathrm{d}\bar{z}' = \mathrm{e}^{\lambda \bar{z}} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\partial_{\bar{z}}^{k-1} g(\lambda, z, \bar{z})}{\lambda^k}, \tag{A.46}$$

where the functions $f(\lambda, z', \bar{z}')$, $g(\lambda, z', \bar{z}')$ are some asymptotic series in λ

$$f(\lambda, z', \bar{z}') = \sum_{k=0}^{\infty} \frac{f_k(z, \bar{z})}{\lambda^k}, \qquad g(\lambda, z', \bar{z}') = \sum_{k=0}^{\infty} \frac{g_k(z, \bar{z})}{\lambda^k} \tag{A.47}$$

such that

$$\partial_{\bar z}(e^{\lambda\bar z} f(\lambda, z', \bar z')) = \partial_z(e^{\lambda\bar z} g(\lambda, z', \bar z')) \tag{A.48}$$

and all expansion coefficients $f_k(z, \bar z)$, $g_k(z, \bar z)$ are smooth functions, all derivatives of these functions are bounded in the z-plane.

The proof of the formula (A.46) is standard and we do not reproduce it here.

(5) The last statement of the Lemma follows directly from the transformation rules for the Bloch functions. □

Proof of Statement 4. Assume, for definiteness, that $\lambda \to \infty_+$. If $\mu \to \infty_+$ then:

$$\omega(\lambda, \mu, z, \bar z) = \left[\frac{1}{2\pi \mathrm{i}(\mu - \lambda)} + \sum_{i,j>0} \frac{\omega_{ij}^{(++)}(z, \bar z)}{\lambda^i \mu^j}\right] e^{(\lambda-\mu)\bar z}\, \mathrm{d}\mu. \tag{A.49}$$

From (A.49) it follows, that

$$\begin{aligned} &2\pi \mathrm{i}\, \mathrm{res}|_{\mu=\infty_+} \omega(\lambda, \mu, z, \bar z) f^{(+)}(\mu, z, \bar z) \\ &\quad = \mathrm{res}|_{\mu=\infty_+} \frac{\mathrm{d}\mu}{\mu - \lambda} e^{(\lambda-\mu)\bar z} f^{(+)}(\mu, z, \bar z) + \mathrm{O}\left(\frac{1}{\lambda}\right) e^{\lambda \bar z}. \end{aligned} \tag{A.50}$$

Applying the well-known formula

$$\mathrm{res}|_{\mu=\infty} \frac{\mathrm{d}\mu}{\mu - \lambda} \mu^n = \begin{cases} \lambda^n & n \geq 0 \\ 0 & n < 0 \end{cases}, \tag{A.51}$$

we get the right-hand side of (A.40).

If $\mu \to \infty_-$, then using (A.34), we get:

$$\omega(\lambda, \mu, z, \bar z) = \left[\sum_{i,j>0} \frac{\omega_{ij}^{(+-)}(z, \bar z)}{\lambda^i \mu^j}\right] e^{\lambda \bar z - \mu z}\, \mathrm{d}\mu. \tag{A.52}$$

Formula (A.44) follows automatically from from (A.52).

Formulas (A.43), (A.41) are proved exactly in the same way. □

Remark 5. A formula, representing the Cauchy kernel on a Riemann surface as a semi-infinite sum of quadratic combinations of eigenfunctions first arose in the article [11] by Krichever and Novikov. In [11] the spatial variable x was discrete. In [6] the spatial variable was continuous, and the CBA kernel was defined as an integral similar to (A.17).

Remark 6. A form, similar to $\tilde\omega(\lambda, \mu, z, \bar z)$, but with integration path, starting from a finite point Z in the z-plane, arises in the theory of finite Darboux transformations (see [13, formula (6.1.18)]). In fact, the same object arose in the generalized Weierstrass map. We do not discuss this analogy in our text, but it seems possible that this analogy has some deep implications.

A.3. Modified Novikov–Veselov equations

Lemma 4. *Let μ be a generic point in Γ. Define a deformation $\delta^{(\mu)}$ of the function $\psi(\lambda, z, \bar{z})$ associated with this point by:*

$$\begin{aligned}\delta^{(\mu)}\psi_1(\lambda, z, \bar{z}) &= \tilde{\omega}(\lambda, \mu, z, \bar{z})\psi_1(\mu, z, \bar{z}) + \alpha(\lambda, \mu)\psi_1(\lambda, z, \bar{z}),\\ \delta^{(\mu)}\psi_2(\lambda, z, \bar{z}) &= \tilde{\omega}(\lambda, \mu, z, \bar{z})\psi_2(\mu, z, \bar{z}) + \alpha(\lambda, \mu)\psi_2(\lambda, z, \bar{z}),\end{aligned} \tag{A.53}$$

where $\alpha(\lambda, \mu)$ is a meromorphic function in λ, uniquely fixed by the following requirement

$$\delta^{(\mu)}\psi_1(\lambda, z, \bar{z}) + \delta^{(\mu)}\psi_2(\lambda, z, \bar{z})|_{z=z_1} = 0. \tag{A.54}$$

Then, (A.53) is a deformation of the Bloch function corresponding to the following deformation of the Dirac operator

$$\delta^{(\mu)}L = \begin{bmatrix} 0 & -\delta_1^{(\mu)}U(z, \bar{z}) \\ \delta_2^{(\mu)}U(z, \bar{z}) & 0 \end{bmatrix}, \tag{A.55}$$

where

$$\begin{aligned}\delta_1^{(\mu)}U(z, \bar{z}) &= \psi_1(\mu, z, \bar{z})\psi_2(\sigma\mu, z, \bar{z}),\\ \delta_2^{(\mu)}U(z, \bar{z}) &= \psi_2(\mu, z, \bar{z})\psi_1(\sigma\mu, z, \bar{z}).\end{aligned} \tag{A.56}$$

For generic μ, these deformations result in non-self-adjoint Dirac operators ($\delta_1^{(\mu)}U(z, \bar{z}) \neq \delta_2^{(\mu)}U(z, \bar{z})$) with complex-valued potentials. But the the Bloch function and the Bloch variety are well defined for such Dirac operators.

Proof of Lemma 4. A simple direct calculation shows that

$$(L + \epsilon\delta^{(\mu)}L)(\psi(\lambda, z, \bar{z}) + \epsilon\delta^{(\mu)}\psi(\lambda, z, \bar{z})) = \mathrm{O}(\epsilon^2). \tag{A.57}$$

Thus, (A.53) defines deformations of the eigenfunction corresponding to (A.55). From (A.38), it follows that the new eigenfunctions satisfy (11) with the same multipliers $w_1(\lambda)$, $w_2(\lambda)$ as the old ones. Thus, the new eigenfunctions are defined on the same curve Γ and have the same periodicity properties. The last property plays a key role when we prove in the following that the functionals $h_{2k+1}[U]$ are invariant under some deformations.

The deformations of the Dirac operator, generated by all $\delta^{(\mu)}$, form a linear space. Deformations preserving the class of self-adjoint Dirac operators with real potentials are the most interesting ones. Let us check that the subspace of such deformations is sufficiently large. □

Lemma 5. *Denote by $\Delta^{(\mu)}$ the following linear combination of deformations $\delta^{(\mu)}$:*

$$\Delta^{(\mu)} = \frac{\delta^{(\mu)} + \delta^{(\sigma\mu)}}{\langle\tilde{\omega}_x(\mu, \mu, z, \bar{z})\rangle_x} + \frac{\delta^{(\theta\mu)} + \delta^{(\sigma\theta\mu)}}{\langle\tilde{\omega}_x(\theta\mu, \theta\mu, z, \bar{z})\rangle_x} \tag{A.58}$$

(recall that $\langle\tilde{\omega}_x(\mu, \mu, z, \bar{z})\rangle_x$ is an even function in μ (see (A.30)) and does not depend on z, $\bar{z}$.)

Then, $\Delta^{(\mu)}$ acts on the space of self-adjoint Dirac operators with real potentials, i.e.

$$\begin{aligned}\Delta_1^{(\mu)}U(z,\bar z) &= \Delta_2^{(\mu)}U(z,\bar z)\\ &= \overline{\Delta_1^{(\mu)}U(z,\bar z)\Delta_2^{(\mu)}U(z,\bar z)}\\ &= \frac{\psi_1(\mu,z,\bar z)\psi_2(\sigma\mu,z,\bar z)+\psi_1(\sigma\mu,z,\bar z)\psi_2(\mu,z,\bar z)}{\langle\tilde\omega_x(\mu,\mu,z,\bar z)\rangle_x}\\ &\quad+\frac{\psi_1(\theta\mu,z,\bar z)\psi_2(\sigma\theta\mu,z,\bar z)+\psi_1(\sigma\theta\mu,z,\bar z)\psi_2(\theta\mu,z,\bar z)}{\langle\tilde\omega_x(\theta\mu,\theta\mu,z,\bar z)\rangle_x}\end{aligned} \tag{A.59}$$

and the Bloch variety $\Gamma[U]$ defined in Section 3 is invariant under these deformations.

Proof of Lemma 5. Formula (A.59) follows directly from (A.56) and (A.58). The Bloch function $\psi(\lambda,z,\bar z)$, for real $U(z,\bar z)$, has symmetry property (17), thus the right-hand side of (A.59) is real. As we pointed out in the previous Lemma, all deformations generated by $\delta^{(\mu)}$ do not change the Bloch variety $\Gamma[U]$ and the multipliers w_1, w_2. This completes the proof. If Statement 2 is fulfilled, then these flows do not change $h_{2k+1}[U]$. □

Eqs. (A.59) are essentially non-local and rather complicated; namely, the right-hand side is expressed in terms of the Bloch function, and it is difficult to calculate Bloch solutions either analytically or numerically. Fortunately, the space of deformations generated by all $\Delta^{(\mu)}$ contains simpler equations such that the right-hand side can be expressed via $U(z,\bar z)$ in terms of quadratures.

Let $\mu\to\infty$. If Statement 2 is fulfilled, we may expand $\Delta^{(\mu)}$ to the following asymptotic series

$$\Delta^{(\mu)}U(z,\bar z) = -2\frac{\mathrm{d}\mu}{\mathrm{i}\mathrm{d}p(\mu)}\sum_{k=0}^{\infty}\frac{K_{2k+1}[U]}{\mu^{2k+2}}+\frac{\overline{K_{2k+1}[U]}}{\bar\mu^{2k+2}}. \tag{A.60}$$

(We write the term $\mathrm{d}\mu/\mathrm{d}p(\mu)$ to gain some standard normalization of the MNV hierarchy. If we omit this multiplier, our expansion coefficients will be linear combinations of Novikov–Veselov generators with constant coefficients, which in most situations is not essential.)

Any $K_{2k+1}[U]$ is a quadratic polynomial of $\phi_l(z,\bar z)$, $\chi_l(z,\bar z)$, $l=1,\ldots,2k$, with coefficients depending on the h_{2l-1} and c_{2l}, $l=1,\ldots,k$, where the c_{2l} are coefficients of the asymptotic expansion (A.31).

For any odd integer $2k+1$, $k\geq 0$, we have the following pair of flows on the space of real double-periodic functions:

$$\frac{\partial U(z,\bar z,t_{2k+1})}{\partial t_{2k+1}} = 2\,\mathrm{Re}\,K_{2k+1}[U], \tag{A.61}$$

$$\frac{\partial U(z,\bar z,\tilde t_{2k+1})}{\partial \tilde t_{2k+1}} = 2\,\mathrm{Im}\,K_{2k+1}[U]. \tag{A.62}$$

Definition 1. Eqs. (A.61) and (A.62) are called the modified Novikov–Veselov equations (MNV).

Remark 7. To define the MNV flows associated to (A.61), (A.62), it is sufficient to have a formal expansion for the Bloch function and the function $h(\lambda)$. Thus, these flows are well-defined for any smooth potential. But without Statement 2 we cannot prove that they conserve the functional $h_{2k+1}[U]$.

Using this definition of the MNV hierarchy, we may immediately prove Statement 3. All functionals $h_{2k+1}[U]$ defined in Section 3 are integrals of motion for all flows $\Delta^{(\mu)}$, thus they are the laws of conservation for their expansion coefficients.

The representation (A.61) may look rather unusual. To check that our definition of the MNV equations coincides with the standard one, let us calculate the deformation of the Bloch function, corresponding to the flows (A.61). To gain a standard answer, we shall use a normalization of the Bloch function different from the one used earlier. Instead of (A.54), we assume that the function $\alpha(\lambda, \mu)$ in (A.53) is identically zero.

Statement 5. *Let the Statement 2 be valid. Then the deformation of the Bloch function corresponding to the flows (A.61) has the following representations:*

$$\frac{\partial \psi(\lambda, z, \bar{z}, t_{2k+1})}{\partial t_{2k+1}}$$
$$= 2\pi \mathrm{i}\{\mathrm{res}|_{\mu=\infty_+} + \mathrm{res}|_{\mu=\infty_-}\}\omega(\lambda, \mu, z, \bar{z}, t_{2k+1},)\mu^{2k+1}\psi(\mu, z, \bar{z}, t_{2k+1}) \tag{A.63}$$

$$= \lambda^{2k+1}\psi(\lambda, z, \bar{z}, t_{2k+1}) + \begin{cases} \mathrm{O}\left(\dfrac{1}{\lambda}\right)\mathrm{e}^{\lambda\bar{z}} & as\ \lambda \to \infty_+ \\ \mathrm{O}\left(\dfrac{1}{\lambda}\right)\mathrm{e}^{\lambda z} & as\ \lambda \to \infty_- \end{cases} \tag{A.64}$$

$$= \left\{\partial_z^{2k+1} + \partial_{\bar{z}}^{2k+1} + \sum_{l=0}^{2k} W_l(z, \bar{z})\partial_z^l + \overline{W_l(z, \bar{z})}\partial_{\bar{z}}^l\right\}\psi(\lambda, z, \bar{z}, t_{2k+1}). \tag{A.65}$$

Proof of Statement 5 (given the validity of Statement 2). By definition

$$-\frac{\mathrm{i}\mathrm{d}p(\mu)}{2}\Delta^{(\mu)}\psi(\lambda, z, \bar{z})$$
$$= \pi \mathrm{i}[\omega(\lambda, \mu, z, \bar{z})\psi(\mu, z, \bar{z}) + [\omega(\lambda, -\mu, z, \bar{z})\psi(-\mu, z, \bar{z})]]_{\mu\to\infty_+}$$
$$+ \pi \mathrm{i}[\omega(\lambda, \bar{\mu}, z, \bar{z})\psi(\bar{\mu}, z, \bar{z}) + [\omega(\lambda, -\bar{\mu}, z, \bar{z})\psi(-\bar{\mu}, z, \bar{z})]]_{\mu\to\infty_-} \tag{A.66}$$

(in this formula, we use $\sigma\mu = -\mu, \theta\mu = -\bar{\mu}$).

The residues in (A.63) are exactly the coefficients of the terms $\mathrm{d}\mu/\mu^{2k+2}$ and $\mathrm{d}\bar{\mu}/\bar{\mu}^{2k+2}$ respectively. Thus, these coefficients give us the action of $K_{2k+1}[U]$ and $\overline{K_{2k+1}[U]}$ on the Bloch function.

Formula (A.64) follows directly from (A.63) and formulas (A.40)–(A.44).

To prove (A.65) let us substitute the asymptotic expansions (A.34), (A.36) in (A.63). We get:

$$\frac{\partial\psi_1(\lambda,z,\bar z,t_{2k+1})}{\partial t_{2k+1}}$$
$$= \left\{R^{(+)}_{2k+1}[\partial_{\bar z}] + \sum_{l=0}^{2k}\phi_{2k+1-l}(z,\bar z)R^{(+)}_l[\partial_{\bar z}]\right\}\psi_1(\lambda,z,\bar z,t_{2k+1})$$
$$+\left\{\sum_{l=0}^{2k}\chi^{(-)}_{2k+1-l}(z,\bar z)R^{(-)}_l[\partial_z]\right\}\psi_2(\lambda,z,\bar z,t_{2k+1}) \quad \text{(A.67)}$$
$$= \partial_{\bar z}^{2k+1}\psi_1(\lambda,z,\bar z,t_{2k+1}) + U(z,\bar z)\partial_z^{2k}\psi_2(\lambda,z,\bar z,t_{2k+1})$$
$$+ \text{ lower order terms} \quad \text{(A.68)}$$
$$= \{\partial_{\bar z}^{2k+1} + \partial_z^{2k+1}\}\psi_1(\lambda,z,\bar z,t_{2k+1}) + \text{ lower order terms} \quad \text{(A.69)}$$

(the $\chi^{(-)}_l(z,\bar z)$ denote the expansion coefficients of the function $\psi_1(\lambda,z,\bar z)$ at the point $\lambda=\infty_-$). A similar calculation shows the following relation:

$$\frac{\partial\psi_2(\lambda,z,\bar z,t_{2k+1})}{\partial t_{2k+1}} = \{\partial_{\bar z}^{2k+1} + \partial_z^{2k+1}\}\psi_2(\lambda,z,\bar z,t_{2k+1}) + \text{ lower order terms.} \quad \text{(A.70)}$$

Statement 5 is proved. □

From (A.65) it follows that the MNV equations are the compatibility conditions for a pair of differential operators on the space of zero eigenfunctions of L. It is well known in soliton theory that such compatibility conditions are equivalent to the existence of standard $L-A-B$ representations (see [2]).

Remark 8. In this article, we constructed a soliton hierarchy in terms of the Cauchy–Baker–Akhiezer kernel.

Using the CBA kernel we construct, in fact, a much wider hierarchy, including essentially non-local equations. All these equations preserve the spectral curve. In Section 4, we show that in fact the deformations corresponding to conformal transformations of the Euclidean space, lie in this wider hierarchy.

In terms of $U(z,\bar z)$, this hierarchy looks rather unnatural. But we may simultaneously treat it as a system of differential equations on a bigger collection of functions; namely we may consider the potential $U(z,\bar z)$ and the wave function in a finite number of fixed points on the spectral curve as unknown functions, connected by the Dirac equations. Similar systems associated with one-dimensional soliton equations were discussed in the literature (see [14] and references therein) from both a mathematical and a physical point of view. In

[15] it was shown, that starting from the KP equation we get a hierarchy naturally containing many other well-known soliton systems.

Acknowledgements

The authors are grateful to I.A. Taimanov for interesting discussions.

References

[1] A.I. Bobenko, Surfaces in terms of 2×2 matrices. Old and new integrable cases, in: *Harmonic Maps and Integrable Systems*, ed. A.P. Fordy (Vieweg, 1994) pp. 83–127.

[2] L.V. Bogdanov, Veselov–Novikov equation as a natural two-dimensional generalization of the Korteweg-de Vries equation, *Theoret. Math. Phys.* 70 (1987) 219–223.

[3] B.A. Dubrovin, A.T. Fomenko and S.P. Novikov, *Modern Geometry – Methods and Applications, Part 1, The Geometry of Surfaces, Transformation Groups and Fields* (Springer, Berlin, 1984).

[4] J. Feldman, H. Knörrer and E. Trubowitz, Riemann surfaces of infinite genus, Part III, preprint ETH, Zürich.

[5] E.V. Ferapontov, Laplace transformations of hydrodynamic type systems in Riemann invariants, *Teor. Mat. Fiz.* 110 (1997) 86–97.

[6] P.G. Grinevich and A.Yu. Orlov, Virasoro action on Riemann surfaces, Grassmannians, det $\bar{\partial}$ and Segal–Wilson τ-function, in: *Problems of Modern Quantum Field Theory*, eds. A.A. Belavin, A.U. Klimyk and A.B. Zamolodchikov (Springer, 1989) pp. 86–106.

[7] P.G. Grinevich, A.Yu. Orlov and E.I. Schulman, On the symmetries of the integrable systems. in: *Important Developments in Soliton Theory*, eds. A. Fokas and V.E. Zakharov (Springer, Berlin, Heidelberg, New York, 1993) pp. 283–301.

[8] G. Kamberov, F. Pedit and U. Pinkall, Bonnet pairs and isothermic surfaces, Duke Math. J., preprint (dg-ga/9511005), to appear.

[9] B.G. Konopelchenko and I.A. Taimanov, Generalized Weierstrass formulae, soliton equations and Willmore surfaces I. Tori of revolution and the mKdV equation, preprint, Ruhr-Universität, Bochum, 1995.

[10] I.M. Krichever, Spectral theory of two-dimensional periodic operators and its applications, *Russ. Math. Surv.* 44 (2) (1989) 145–225; Perturbation theory in periodic problems for the two-dimensional integrable systems. Sov. Sci. Rev., Math. Phys. Rev. (Gordon and Breach, Harwood Academic Publishers, 1992) p. 9.

[11] I.M. Krichever and S.P. Novikov, Virasoro-type algebras, Riemann surfaces and strings in Minkovsky space, Funct. Anal. Appl. 21 (1987) 294–307.

[12] P. Kuchment, *Floquet Theory for Partial Differential Equations* (Birkhäuser, 1993).

[13] V.B. Matveev and M.A. Salle, *Darboux Transformations and Solitons* (Springer, 1991).

[14] V.K. Melnikov, New method for deriving nonlinear integrable systems, J. Math. Phys. 31 (5) (1990) 1106–1113.

[15] A.Yu. Orlov, Volterra operators algebra for zero curvature representation, Universality of KP, in: *Nonlinear (and turbulent) Processes in Physics*, eds. A. Fokas, D. Kaup, A. Newell and V.E. Zakharov (Springer seria in nonlinear dynamic, 1991) pp. 126–131; A.Yu. Orlov, Symmetries for unifying different soliton systems into a single hierarchy, preprint IINS/Oce-04/03 (March 1991); B. Enriquez, A.Yu. Orlov and V.N. Rubtsov, Dispersionful analogues of Benney's equations and N-wave systems, Inverse Prob. 12 (1996) pp. 241–250.

[16] M.U. Schmidt, Integrable systems and Riemann surfaces of infinite genus, *Mem. Am. Math. Soc.* (1996) 581.

[17] I.A. Taimanov, Modified Novikov–Veselov equation and differential geometry of surfaces, preprint (November 1995) (dg-ga/9511005), to appear in Translations of the American Mathematical Society.

[18] I.A. Taimanov, Surfaces of revolution in terms of solitons (dg-ga/9610013).

[19] A.P. Veselov and S.P. Novikov, Finite-zone, two-dimensional potential Schrödinger operators, Explicit formulas and evolution equations, Sov. Math. Dokl. 30 (1984) 588–591; Finite-zone, two-dimensional potential Schrödinger operators, Potential operators, Sov. Math. Dokl. 30 (1984) 705–708.
[20] T.J. Willmore, *Riemannian Geometry* (Clarendon Press, Oxford, 1993).
[21] V.E. Zakharov, S.V. Manakov, S.P. Novikov and L.P. Pitaevsky, *Soliton Theory* (Plenum, New York, 1984).